The complete solution of Alt-Burmester synthesis problems for four-bar linkages

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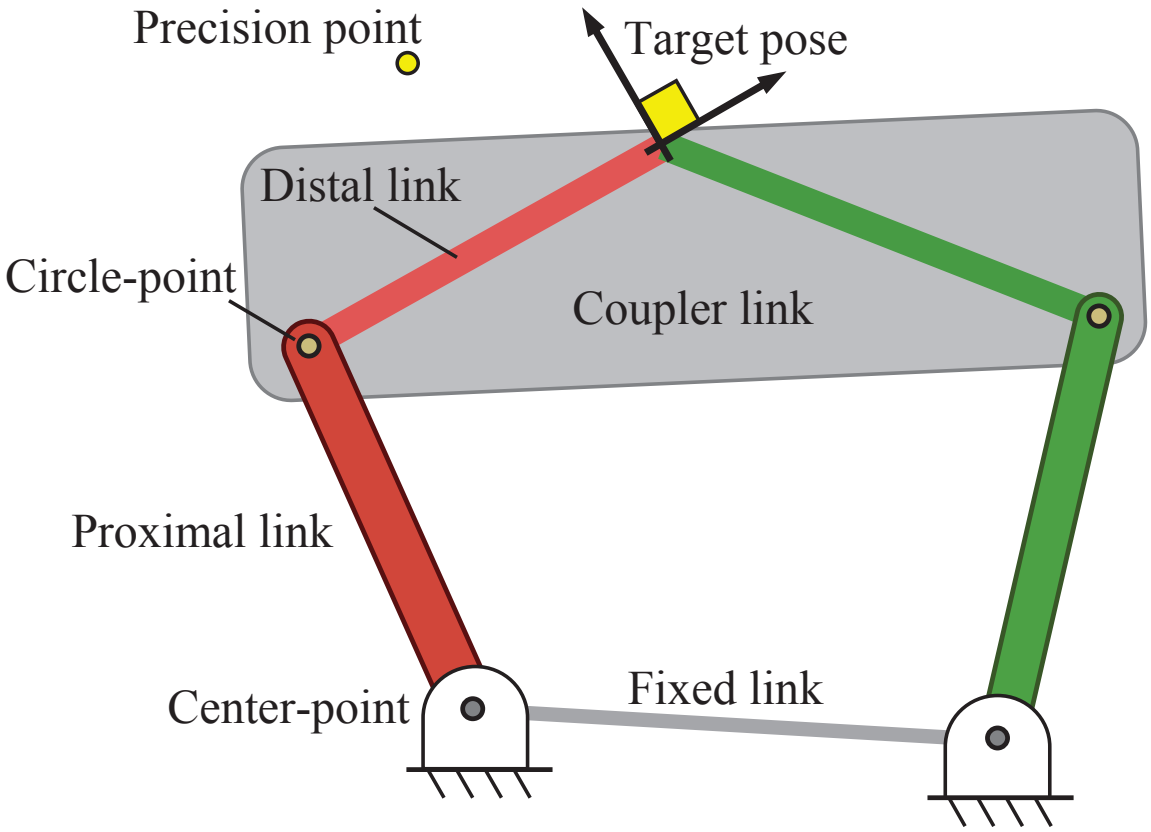
Outline

Four-bar linkages

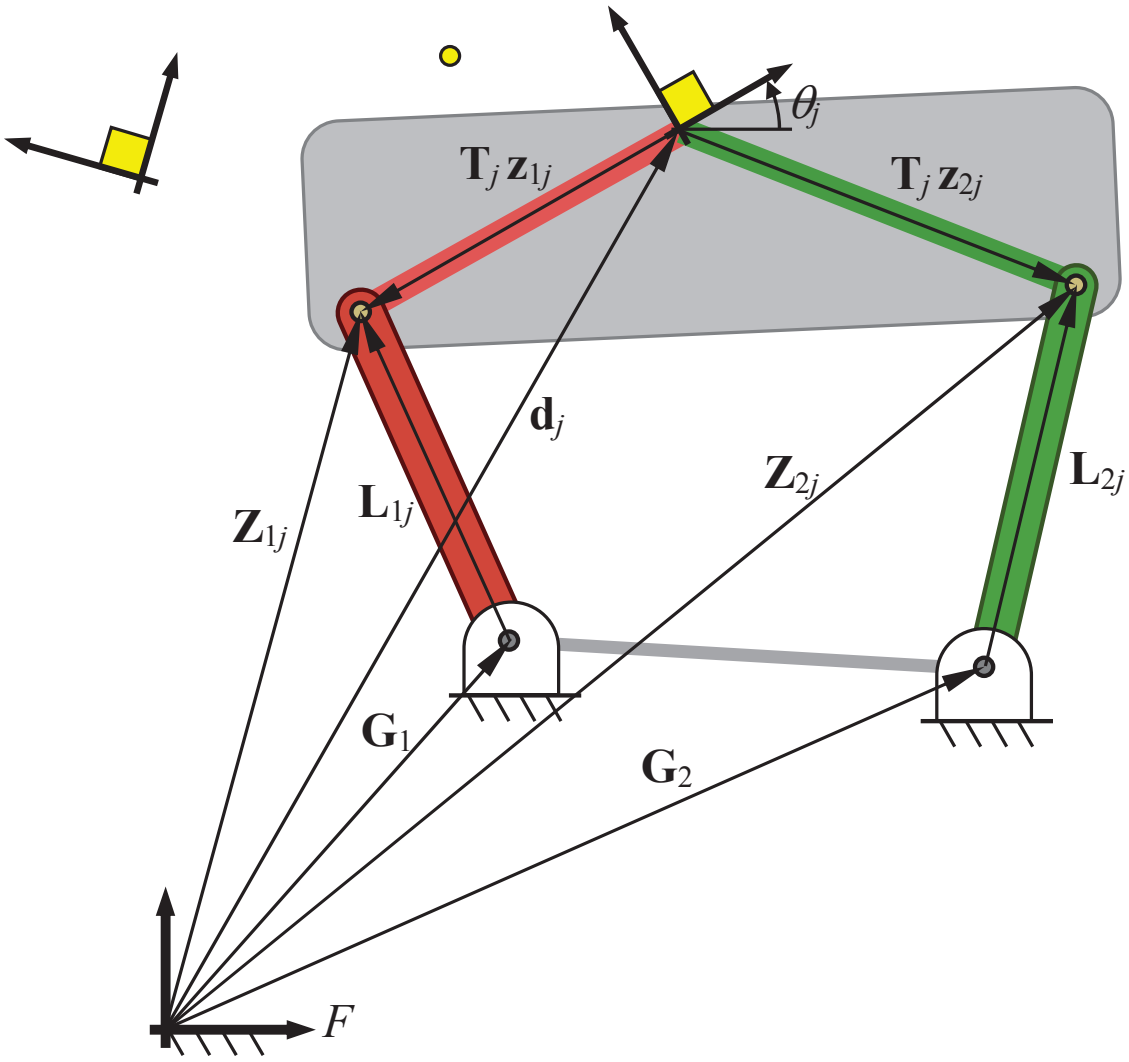


Kinematic Synthesis

Design a mechanism with a given motion:

1. Select particular design constraints representing requirements
2. Phrase as a polynomial system
3. Solve the problem
4. Select a computed mechanism
5. Build it

1. Select design constraints



- $M = \#$ poses
  yellow squares
- $N = \#$ points
  yellow dots

2. Phrase as a polynomial system

for $j = 2, \ldots, M+N$

$$\mathbf{L}_{1j}\,\bar{\mathbf{L}}_{1j} - \mathbf{L}_{11}\,\bar{\mathbf{L}}_{11} = 0. \tag{1}$$

for $j = 1, \ldots, M+N$

$$\mathbf{L}_{2j} = \Theta_j\,\mathbf{z}_2 + \mathbf{D}_j - \mathbf{G}_2, \tag{2}$$
$$\bar{\mathbf{L}}_{2j} = \bar{\Theta}_j\,\bar{\mathbf{z}}_2 + \bar{\mathbf{D}}_j - \bar{\mathbf{G}}_2, \tag{3}$$

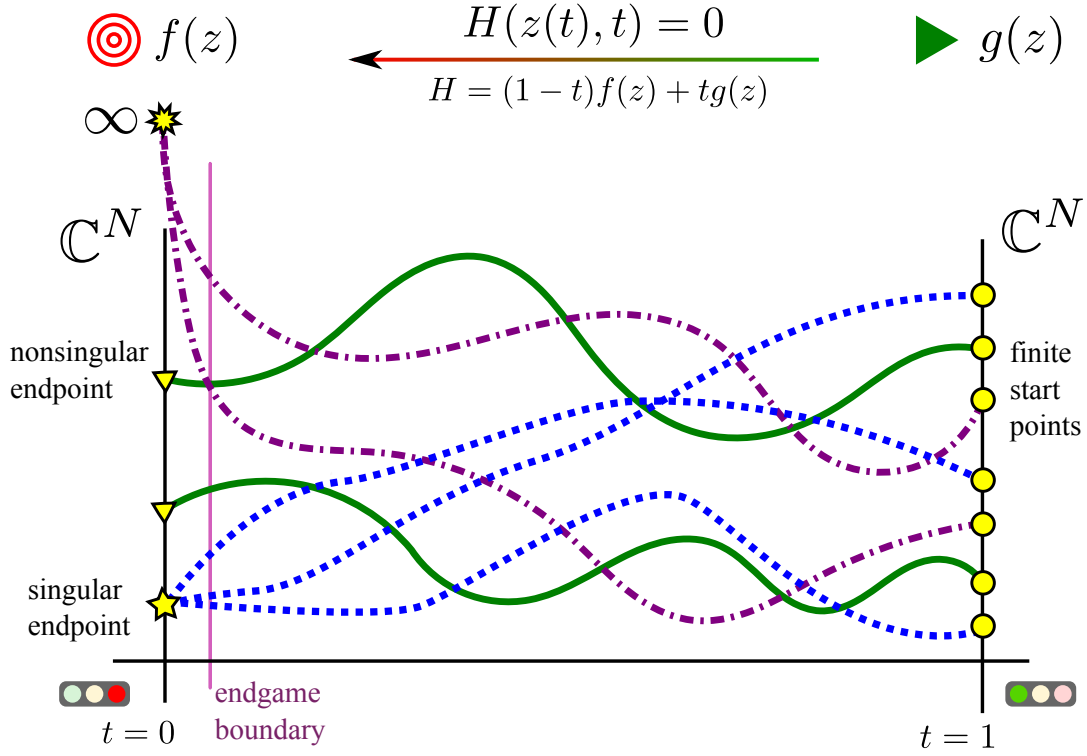
along with, for $j = 2, \ldots, M+N$,

$$\mathbf{L}_{2j}\,\bar{\mathbf{L}}_{2j} - \mathbf{L}_{21}\,\bar{\mathbf{L}}_{21} = 0. \tag{4}$$

Gives "the (M, N) problem"

3. Solve the problem

The main numerical method used to solve polynomial systems is *homotopy continuation*



1. Form a homotopy.
2. Track from start time and point to target.

4. Select computed mechanism

- To build a mechanism, you have to have its design parameters
- The design parameters *are* the solution to the polynomial system
- Depending on M,N, the dimension D of the set varies:

$$D = 10 - 2M - N. \tag{5}$$

- For some M, N, $D = 0 \Rightarrow \exists$ fintely many solutions.
- For others, $D > 0 \Rightarrow \infty$ many solutions

5. Build it!

Now the maker in you comes out. Have design parameters, just need to

- get some bearings
- get some motors
- get a tool for the end effector
- write software to generate 3D model from design parameters to fit the above
- print
- assemble

Outline

Complete solution

What do I mean by *complete solution*?

- For each possible (M, N) problem describe
  - The dimension of the set
  - The degree of a generic solution set, under two projections
- and additionally, for some (M, N),
  - The number of critical points of 1-dimensional sets

Degrees

Table: Dimension and degree of Alt-Burmester solution sets, projected onto all eight natural variables.

		M (poses)									
		1		2		3		4		5	
N	0	8	1	6	4	4	16	2	16	0	16
	1	7	7	5	24	3	64	1	48		
	2	6	43	4	134	2	194	0	60		
	3	5	234	3	552	1	362				
	4	4	1108	2	1554	0	402				
	5	3	3832	1	2388						
	6	2	8716	0	2224						
	7	1	10858								
	8	0	8652								

Table: Dimension and degree of Alt-Burmester solution sets, projected onto the center point coordinates, $(\mathbf{G}_1, \overline{\mathbf{G}}_1)$.

		M = 1	2	3	4	5
N	0	*	*	*	1 3	0 4
	1	*	*	*	1 3	
	2	*	*	*	0 60	
	3	*	*	1 128		
	4	*	*	0 402		
	5	*	1 816			
	6	*	0 2224			
	7	1 3500				
	8	0 4326				

* indicates closure of projection filled entire space

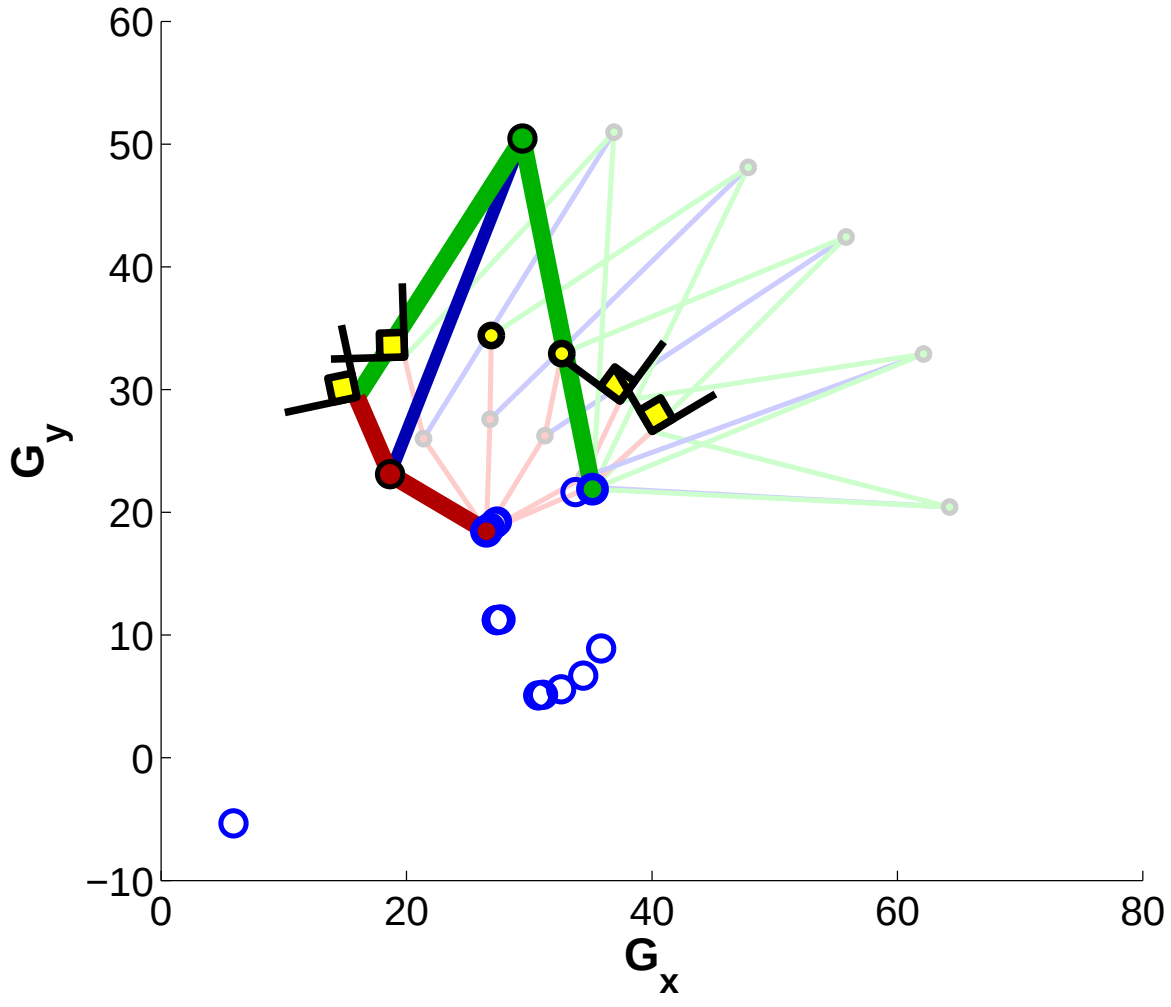
Numbers of critical points

Table: Numbers of critical points for all one-dimensional Alt-Burmester problems.

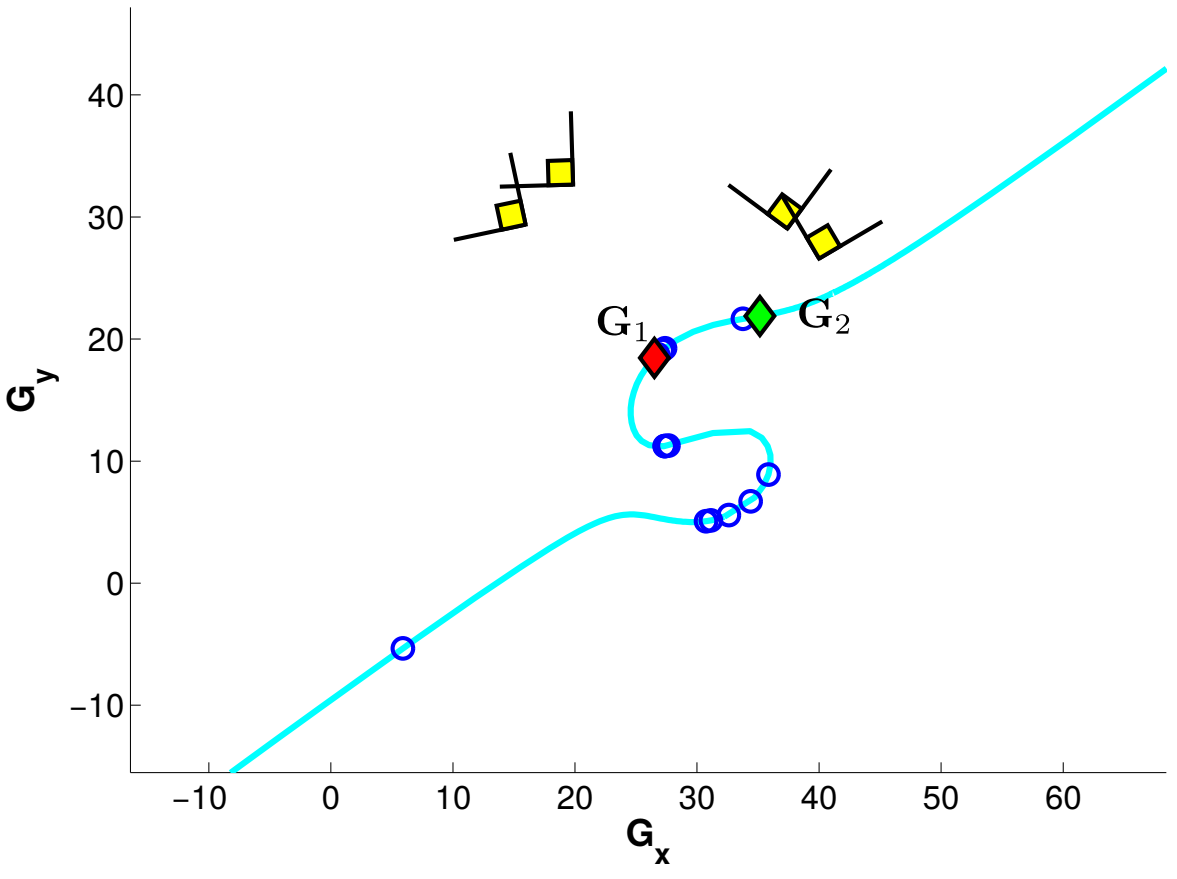
(M, N)	Generic number of critical points					
	8 natural vars		$(\mathbf{G}_1, \overline{\mathbf{G}}_1)$		$(\mathbf{z}_1, \overline{\mathbf{z}}_1)$	
$(1, 7)$	55676	4168	38740	4168	44208	4168
$(2, 5)$	11228	988	8084	988	8456	988
$(3, 3)$	1440	144	972	144	1000	144
$(4, 1)$	152	16	92	16	92	16
	nonsing.	sing.	nonsing.	sing.	nonsing.	sing.

Outline

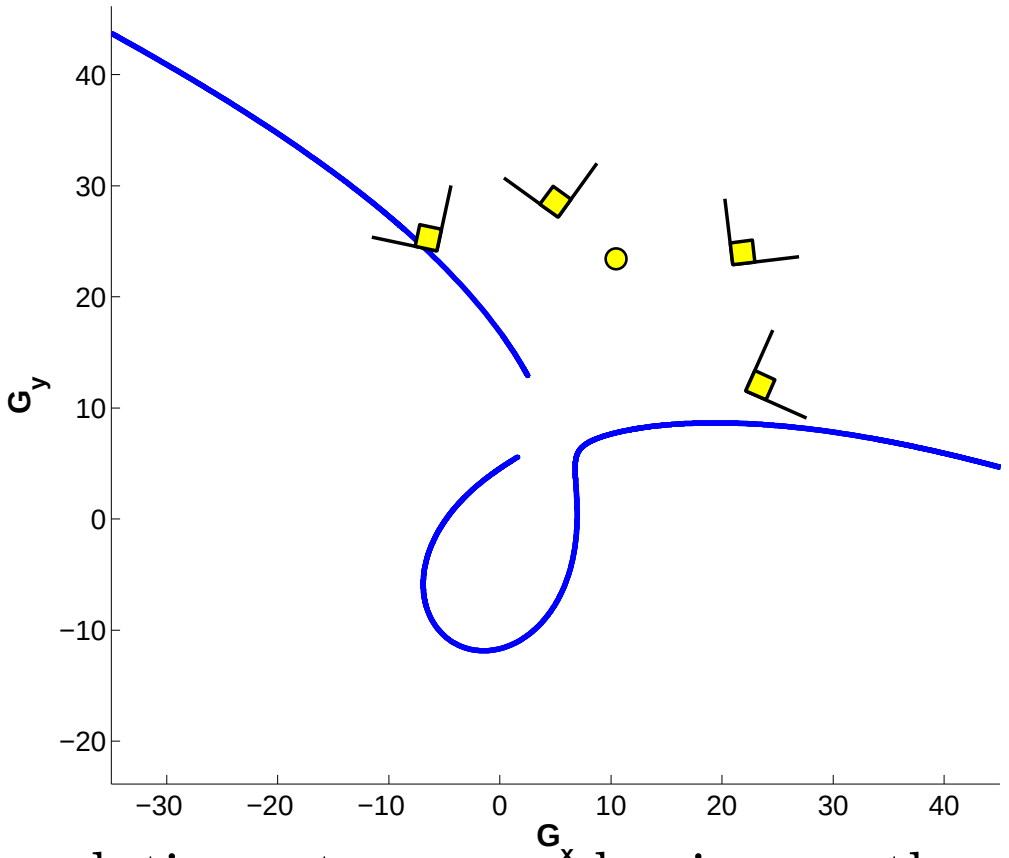
A (4,2) problem – point solutions



Previous (4,2) solution, on a curve of (4,0) solutions

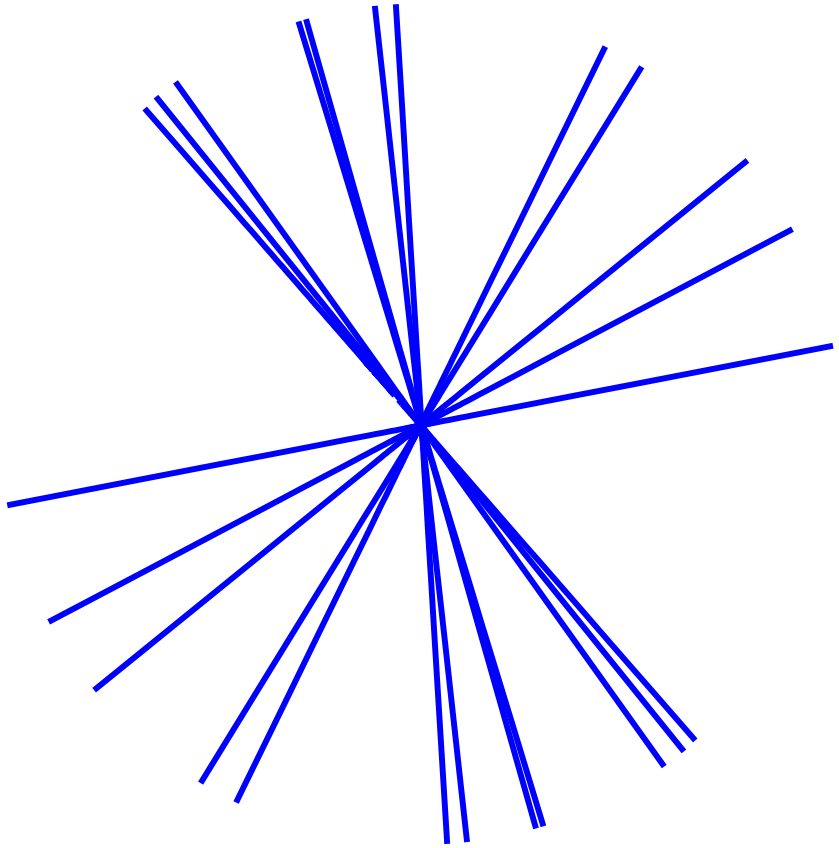


A (4,1) problem – a curve of solutions



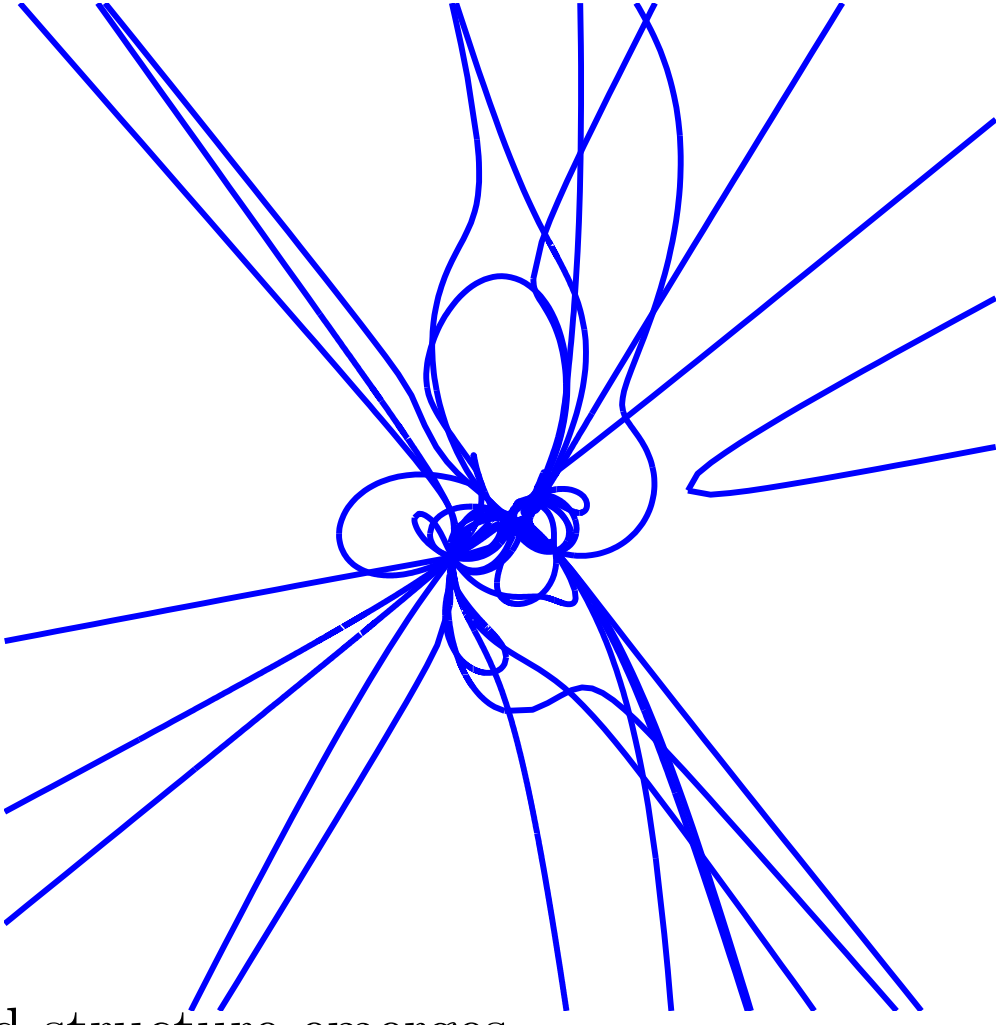
discontinuous solution set – no mechanism on the missing section satisfies the constraints

A (3,3) problem – a curve of solutions



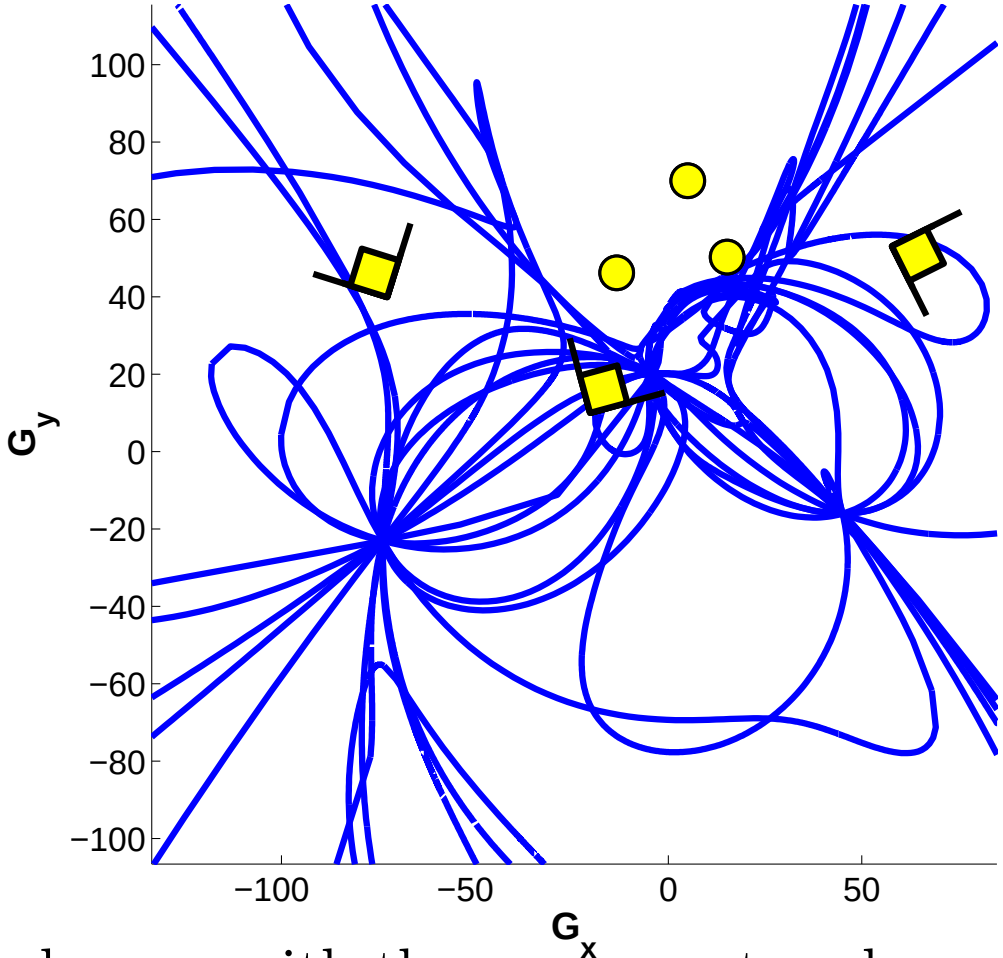
12 asymptotes to infinity – correspond to slider mechanisms

A (3,3) problem – zoom 1



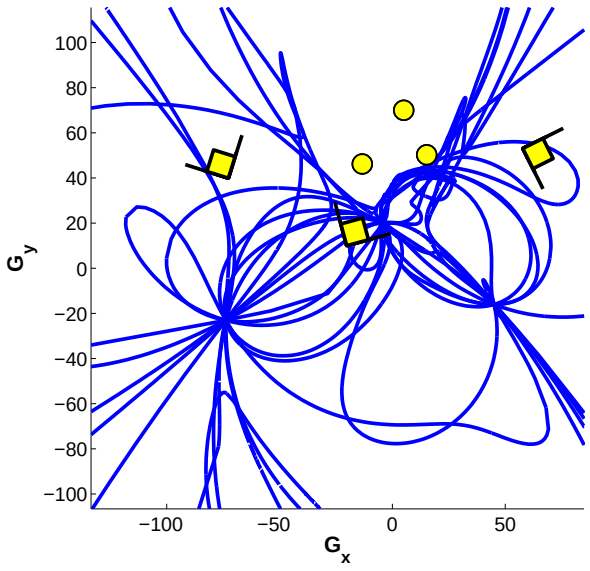
a complicated structure emerges

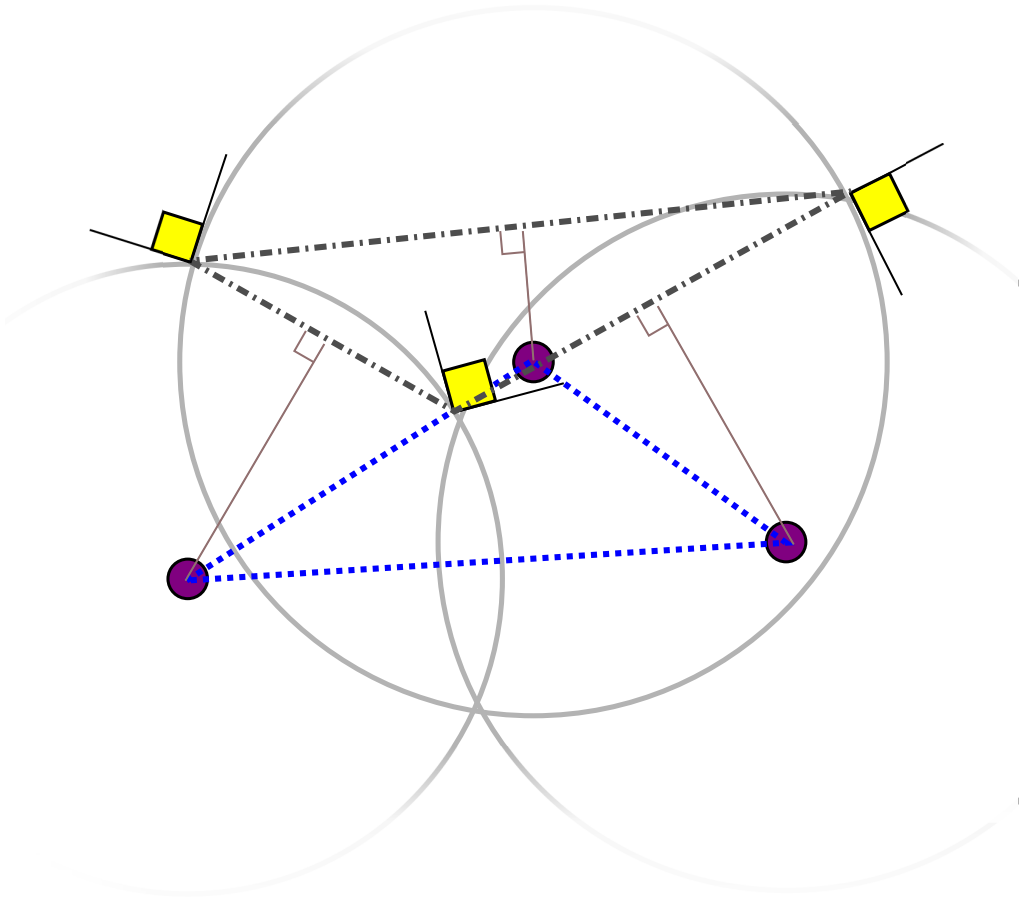
A (3,3) problem – zoom 2



what a tangled mess, with three apparent nodes

A (3,3) problem – the pole triangle





The nodes are on the vertices of the pole triangle!

Outline

Conclusions

The problem of four-bar linkage kinematic synthesis is solved

- Degree and dimension for solution set for all (M, N) problems
- How many critical points there can be for curve solutions
- How to compute a mechanism for any (M, N) problem

Future work:

- Tackle spatial mechanisms
- automatic 3D-printing of a target mechanism
- Optimization on positive-dimensional solution sets

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